CHAPTER 9

Geometry and Spatial Sense

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Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world (28).

What comes to mind when you think of "geometry"? What might your students think? Responses would reflect the respective geometry learning experiences of both you and your students. This chapter is intended to familiarize you with current research on learning and teaching geometry in the primary grades and to help you use research-based ideas to enrich your students' geometry experiences. Before reading on, think about the geometry learning environment in your classroom. What geometry is taught, and how do you present it? What geometric understandings and spatial abilities do youngsters bring to the classroom, and how does your instruction enhance your students' geometric thinking? We will review related research to provide some perspectives for addressing these and other questions about learning and teaching geometry.

Geometry in K–4: Goals and Content

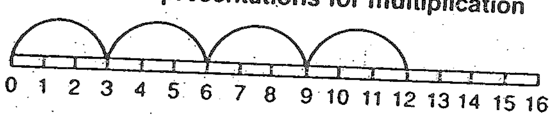
Is geometry an important part of your students' mathematics program? National, state, and local K–12 curriculum guidelines recommend that geometry play a prominent role in school mathematics. NCTM's *Curriculum and Evaluation Standards for School Mathematics (Standards)* (28) states that for grades K–4: "the mathematics curriculum should include two- and three-dimensional geometry so that students can—

- describe, model, draw, and classify shapes;
- investigate and predict the results of combining, subdividing, and changing shapes;

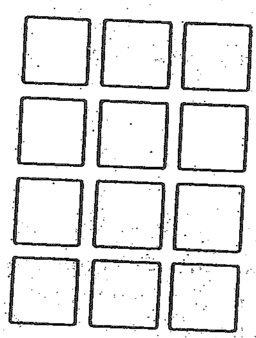
- develop spatial sense;
- relate geometric ideas to number and measurement ideas;
- recognize and appreciate geometry in their world." (28, p. 48)

The K–4 mathematics curriculum contains a host of geometry topics. We may recognize some topics as ones we learned in elementary school. Others may be less familiar: line of symmetry; topological ideas (open/closed curves); motion geometry such as slides, flips, turns; coordinates (see Fig. 9.1); and Logo geometry. Geometry also plays an important role in many nongeometric topics. Children can represent multiplication as repeated addition on a number line, or by an array of squares that connects it with area. They can use the geometry of familiar shapes to build an understanding of fractions (3, 4, 14) and notions of probability. In bar graphs children use their understanding of size to communicate about number. Inadequate geometric understanding can adversely affect learning of these other topics.

Geometric representations for multiplication



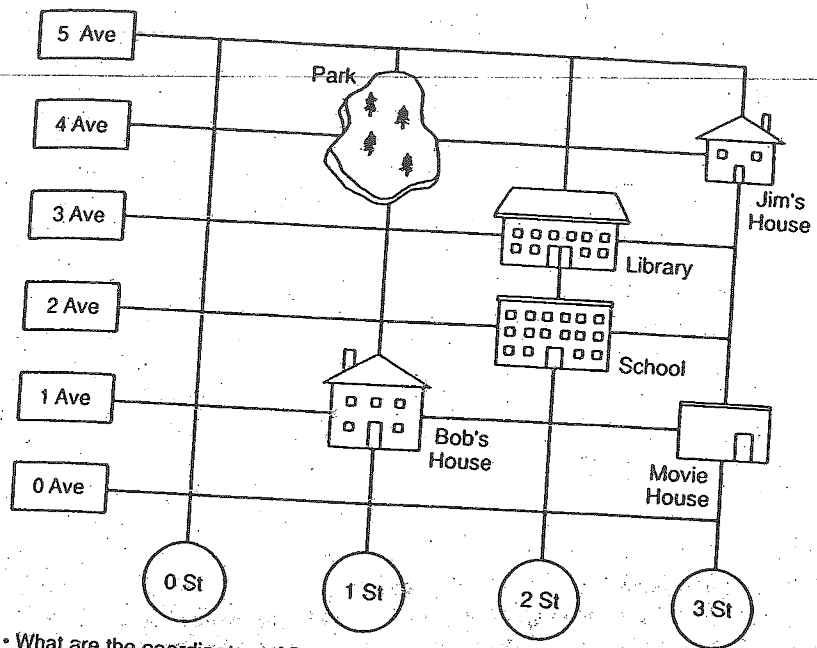
- 4 × 3 on a number line.



- 4 × 3 as an array of tiles. How many did you use?
- Make other rectangle arrays with the 12 tiles, and write a multiplication sentence for each.
- Explore: What rectangle arrays could you make with 11 tiles? 24?
- *Research idea:* Investigate how your students use geometric representations to explain thinking strategies for facts and to solve problems.

Geometry also provides a rich context for the early development of mathematical thinking, from lower-order thinking processes such as identifying shapes to higher-order processes such as discovering properties of shapes, creating geometric patterns, and solving geometric puzzles and problems in different ways (5, 11, 14, 43). Thus we can use geometry to implement the four NCTM standards related to process: *mathematics as problems solving, mathematics as communication, mathematics as*

FIGURE 9.1 Coordinate geometry and paths (5, 38)



- What are the coordinates of Bob's house? Jim's house? the park?
- Find one way to go from Bob's house past the library to the park. How many blocks long was this path?

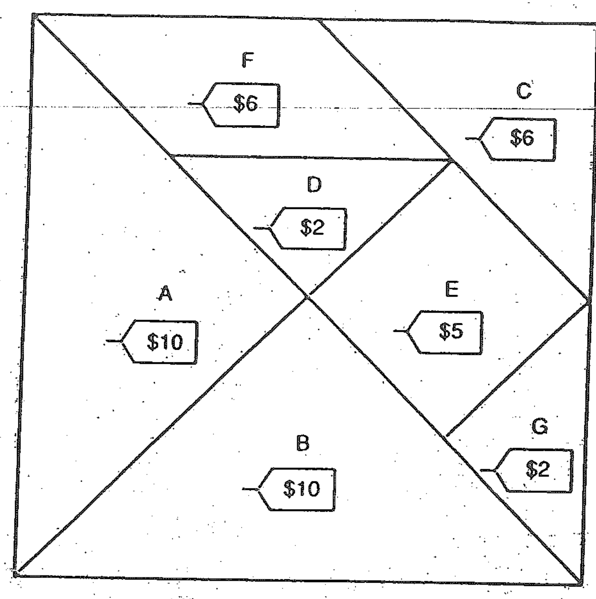
Explore: Find other paths. Are they all the same length? What are the shortest?

reasoning, and *mathematical connections*. We need to craft activities that feature several processes and interrelate geometry with other areas (Fig. 9.2). Geometry topics should not be taught in isolated units, but rather should be a natural and an integral part of the entire curriculum.

Although recommendations call for a rich geometry program in K–4, research indicates that, regrettably, little geometry is taught in the elementary grades, and that what is taught is often feeble in content and quality (5, 13, 31). When taught, geometry was the topic most frequently identified as being merely taught for "exposure," that is, given only brief, cursory coverage (31). Now, as implementation of the *Standards* goes forward, we need to examine what geometry we actually teach and take steps to close any gap between the recommended geometry curriculum and the one we provide our students.

We now examine five research perspectives that can help us implement an enriched geometry curriculum in the primary grades. The first two focus on how young children develop spatial thinking, the third on levels at which children can reason in geometry, and the fourth on concept learning. The fifth perspective involves more general research in early childhood education. Activities and suggestions are provided throughout the chapter to illustrate how we can apply knowledge gained from re-

FIGURE 9.2 Problem solving with tangrams



Cut out the tangram pieces and solve these problems.

- Use D and ___ to make a $4 triangle. Can you make a $4 square with them?
- Use ___, ___, and ___ to make a $10 rectangle. What other shapes can you make for $10?
- Make a square. How much did it cost?

Explore: Make other squares. What do they cost?

- Make a $20 polygon. How many sides? angles?

Explore: How many different $20 polygons can you make?

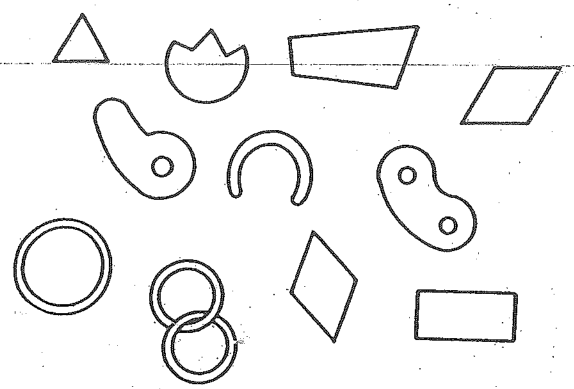
search in our teaching and also how we too can undertake "action research" in our classrooms.

A Piagetian Perspective

How do young children perform on such tasks as copying a triangle, identifying cutout shapes, or drawing how a cardboard box would look if we opened it up and laid it flat on a table? Insights into how children at different ages approach such spatial tasks are provided by the research of Jean Piaget and others. Piaget investigated children's spatial thinking through clinical tasks, such as those below, which involved mental "representation" of objects.

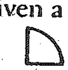
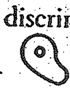
Touch and Draw/Identify. Present the child with real objects or cardboard cutouts; let him or her touch and feel around each without being allowed to see it (Fig. 9.3). Then ask the child to name, draw, or point to it from a collection of visible objects (30, p. 18).

FIGURE 9.3 Shapes for Piagetian touch and draw/identify task



Look/Trace and Draw/Construct. Show drawn figures; let the child trace them with a finger or even guide his or her motions. Without being able to look at the pieces, have the child draw them or copy rectilinear figures, using sticks (30, p. 53).

Look and Draw a Shadow. Show how a light casts a shadow of an object on a screen when the object is placed between a light and the screen. Present objects and have the child predict and draw what its shadow on the screen would look like (30, p. 195).

Piaget found that children's performance on such tasks was developmental in nature, that is, they gradually develop spatial and geometric ideas in a spontaneous, independent, and age-related manner. Moreover, children first deal with *topological* aspects of shapes such as inside/outside, closed/open, then *projective* aspects such as curved/straight-sidedness, and finally *Euclidean* or "size" aspects such as length, angle size, and area. For example, on the touch and draw/identify task above, a young child (ages 3–4), when given a , identified it or another shape that is topologically the same (such as ), but did not discriminate curvilinear shapes from rectilinear ones, perhaps identifying an oval or  when given a square. Children (roughly ages 4–6) were more active in exploring shapes tactily, and now distinguished shapes according to projective aspects but did not distinguish among shapes using Euclidean aspects, perhaps picking a rectangle or triangle when given a diamond. Finally, older children (about ages 6–8) explored shapes methodically, using Euclidean notions, and distinguished among complex forms.

Similar results on other spatial tasks supported the contention that "topological relations universally take priority over Euclidean relations" (30). This progression of spatial thinking from topological to projective and then to the Euclidean, known as the "topological primacy thesis," is the opposite of the one used in most primary school programs, which typically begin with measurement aspects of geometry. Some educators have used this thesis as a rationale for providing "developmentally appropriate" topological and projective experiences before Euclidean ones. They suggest that we give children qualitative activities before quantitive measurement tasks (33, 36) and provide environments that include building blocks (9, 35, 36), shadow-geometry (22, 36), and explorations of shapes drawn on balloons or made with Cuisenaire rods or modeling clay (9, 29, 36).

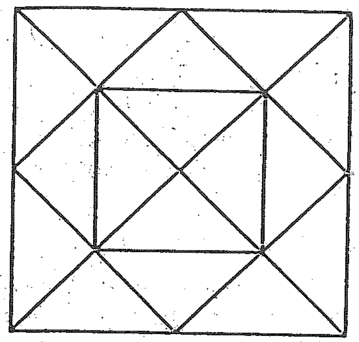
The topological primacy thesis is not uniformly supported by all the research. Some topological concepts such as "inside/outside" were found to develop earlier than others such as order and continuity (23). A study involving over 600 first graders in a program that included topological activities indicated that without studying topology, children could correctly answer about 70 percent of the topology questions posed, and that with instruction there was little improvement (37). Also, children learned Euclidean concepts without instruction on topological ideas. The investigator concluded that early programs in geometry that first deal with topology or that stress "knowledge" of content are not warranted. Rather, teachers should provide activities that sharpen youngsters' geometric thinking. Topological primacy thesis aside, this recommendation is in the spirit of Piaget's constructivist view that children develop spatial concepts by acting on objects and reflecting on their actions, not simply by looking at the objects. In our classes, we can use modifications of Piagetian-type tasks to investigate children's spatial thinking ourselves and also to provide informal instruction that nurtures spatial thinking. Youngsters can try a "feel and tell" grab-bag activity with familiar objects and geometric shapes (9), create block constructions and share what they did with their teacher and classmates (15, 35), or sit in a circle around an object, draw it, and compare drawings. Older children might draw an object showing different perspectives, perhaps as part of their study of modern art and cubism.

> "The first task at the primary school level should not be to give a 'foundation of topological concepts' nor a 'preparation for symmetry' nor [is it] to convey 'knowledge of plane and three-dimensional basic shapes.' Instead an early program of geometry should make use of geometrical activities, mediate a readiness to experiment with paper, scissors, glue, strings, wood, etc., [encourage] a willingness to make sketches and exact drawings, and lead to *thinking* about these activities. In short, a program should initiate geometrical actions, drawing, and thinking." (37, pp. 295–296)

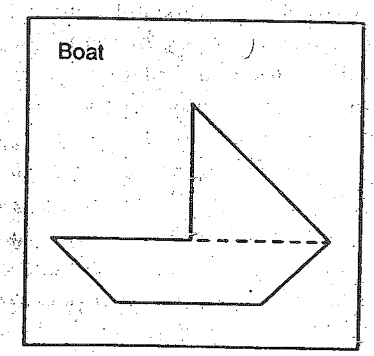
Spatial Sense

How would boys and girls do on spatial tasks such as finding triangles in a design, solving Tangram puzzles, completing a symmetric figure, or replicating a 3-D sculpture made with Cuisenaire rods? (See Fig. 9.4.) Tasks like these involve various aspects of spatial sense such as interpretation of visual information, visual memory, and visual processing such as rotating objects mentally (spatial visualization) and recognizing relationships between various parts of a configuration. Spatial abilities

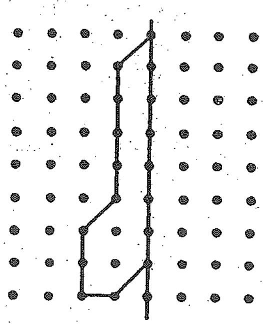
FIGURE 9.4 Spatial thinking tasks



How many triangles can you find?
How did you find them?



Use tangrams. Can you make this in two different ways?



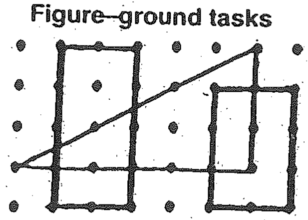
Complete this rocket ship.

and visual imagery play a vital role in mathematical thinking and are important for technical-scientific occupations (7, 12).

> "Spatial sense is an intuitive feel for one's surroundings and the objects in them. To develop spatial sense, children must have many experiences that focus on geometric relationships: the direction, orientation, and perspectives of objects in space, the relative shape and sizes of figures and objects, and how a change in shape relates to a change in size." (28, p. 49)

The development of "spatial sense" in grades K–4 is a clear mandate in the *Standards* (28). This early attention to spatial thinking is supported by research of psychologists such as Piaget, art educators, and mathematics educators who identified developmental stages in children's drawings and other areas of spatial thinking and by research on spatial training for boys and girls. A recent comprehensive summary of the research on gender differences in mathematics and science revealed that, up to the mid-1970s, males outperformed females on many spatial tasks, but since then gender differences in spatial abilities are declining and the differences that remain are responsive to training (21). A close look at such tasks as mentally rotating a figure showed that no gender differences existed on accuracy but some on speed were evident favoring males. However, training reduced these differences. Research now suggests that we should accept spatial abilities as malleable and explore what instructional interventions and experiences can ameliorate performance. This recommendation is further supported by spatial training studies in primary grades using tangrams (45) and via a unit the geometry of slides, flips, and turns (12). Children in primary schools where the use of manipulatives predominated tended to perform better on spatial tasks than children in "material-free" schools (4). Collectively these studies tell us to provide hands-on spatial activities in grades K–4 and to assess their effectiveness for strengthening the spatial thinking of boys and girls. Since both boys and girls were found to vary considerably in accuracy and speed on spatial tasks, we need to be sensitive to time constraints and difficulty of tasks—perhaps presenting activities in a learning center where children can work at their own pace and choose tasks at different levels of complexity.

Figure–ground tasks



- How many rectangles can you find? How many triangles?
- On your geoboard make overlapping rectangles and triangles that make more triangles and rectangles.

Record this on geoboard dot paper.

One research-based suggestion for the classroom is to stress children's use of manipulatives in activities that are designed to foster aspects of spatial perception (5,12) such as *figure-ground* (identifying a specific figure in a picture or complex configuration) and *spatial relationships* (relating the position of two or more objects) (Fig. 9.5).

Another suggestion is to show children shapes, have them construct or draw what they see, and then respond to related questioning ("How did you see this? What did you copy first? Next? Could you copy it another way?"), which further stimulates spatial thinking (45). Second graders who used tangrams or drew diagrams to re-create visual images briefly shown on an overhead projector and then answered related questions showed "dramatic improvements in their spatial imagery over the course of a year" (45, p. 58). Moreover, this helped students "develop geometry concepts and learn geometric vocabulary, find geometric shapes in complex drawings, and develop such spatial operations as rotating images . . . They began to think about visually presented images in more than one way and to elaborate on and extend their own ideas" (45, p. 54).

A variation on the copy-a-shape task is to have children copy designs that encourage them to think about designs both holistically and in terms of its parts (24). Children might outline or color the designs on dot or grid paper (5, 14, 24). Also, a child can make a sculpture with color cubes or Cuisenaire rods and then tell a partner, who cannot see it, how to make it, which also fosters communication via spatial vocabulary. Free play, and lots of it, in spatial environments such as the "block corner" develops spatial thinking and promotes the use of spatial language (2, 15, 35). It also provides a setting for informal assessment of visual-perceptual strengths and weaknesses (12). Other ideas (24) include the use of two- and three-dimensional puzzles, construction of solids with commercial or everyday materials (toothpicks, small marshmallows), math-art projects, and position-in-space games (2) such as Simon Says. For further ideas see the February 1990 *Arithmetic Teacher*, which focuses on spatial sense.

Imagine "happiness." Perhaps the image you have is of an event, a color, or "a happy shape." Spatial thinking can be thought of as a visual form of general "imag-

FIGURE 9.5 Spatial relationships task

Use cubes to make a building like the one shown. (12, p.17)

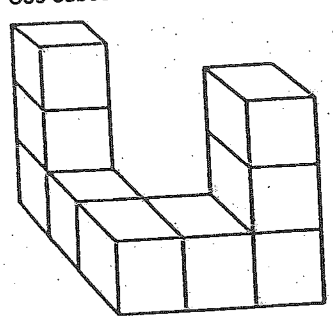
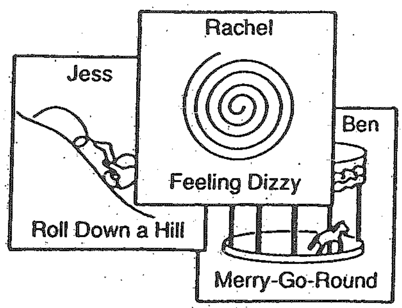


FIGURE 9.6 An imaging activity

Ask children to imagine being a circle. What could you do? How might you feel? Think of a "circular experience." Have children express their ideas by drawing, oral or written stories, or dramatizations.



ing" (Fig. 9.6). The need for instruction on imagery is suggested by a recently developed body of research on imagery and its role in learning, memory, creativity, and problem solving (38). Use of visual imagery has proved effective in solving mathematics problems. Rather than make immediate use of numerical data in word problems, children were asked to close their eyes and imagine the problem situation, draw a picture of it, and then add relevant numerical data to the picture. This approach generated new enthusiasm for mathematics and improved problem-solving performance, especially for children with low reasoning ability (26). Using geometry in this way should be part of every student's repertoire for representing and solving problems and for communicating mathematically.

The van Hiele Model of Geometric Thinking

How does the geometric thinking experienced by students in the primary grades relate to their geometric thinking in the middle grades or even later when they study geometry in high school? This question was explored by two Dutch educators, Dina van Hiele-Geldof and Pierre van Hiele, who were concerned about difficulties many of their students encountered in secondary school geometry. They believed that secondary geometry, which emphasizes deductive reasoning and proof, requires a "high" level of thinking and that their students did not have sufficient prerequisite experiences in thinking at lower levels during the elementary grades. They formulated a model of five "levels of thinking" and proposed instructional "phases" to promote students' progress from one level to the next (44).

The van Hiele Model: Levels and Instructional Phases

According to the van Hieles, the learner, assisted by appropriate instruction, passes through levels of thinking, from visual recognition of shapes by their appearance as a whole (level 0) to analysis and description of shapes in terms of their properties (level 1) to three higher "theoretical" levels involving informal deduction (level 2), then formal deduction involving axioms and theorems (level 3), and finally work with abstract geometric systems (level 4).

Geometry in grades K–4 involves thinking mainly at levels 0 and 1. At level 0 (the *visual* level) a child judges shapes by their appearance as a whole. A square is identified because it "looks like one." P. M. van Hiele says, "There is no why. One just sees it" (44, p. 83). Young children need experiences that develop their global understanding of geometric objects such as constructing and drawing shapes, fitting 2- or 3-D shapes together, and looking for shapes in their home and school environments.

At level 1 (the *descriptive-analytic* level) children think about shapes in terms of their parts and properties involving the parts. Through experimentation (measuring, folding, drawing, or working with models), they discover properties of shapes such as squares have "four sides," "all equal sides," "all right angles." They also use the language of properties to describe shapes and to explain solutions for geometric problems. However, they do not yet deduce certain properties from others or consider which properties are necessary and sufficient for defining a shape. This occurs at level 2 (*informal deduction*) when children begin to use deductive reasoning to establish "why," such as explaining why all squares are rectangles but not all rectangles are squares. (See 10 and 13 for detailed descriptions of the levels and related activities for children.)

The van Hieles ascribed characteristics to the levels—such as being discontinuous and hierarchical with performance at one level requiring success at lower levels. Each level has its own language; for example, "square" on level 0 means a shape that "looks like one," while on level 1 it conveys a shape with certain properties, and on level 2 it is specified by a definition. The van Hieles assert that many failures in teaching geometry result from a communication gap between teacher and students, who have different meanings for geometric language and hence are talking on different levels. Finally, progress within a level and to the next level depends more on instruction than maturation or age.

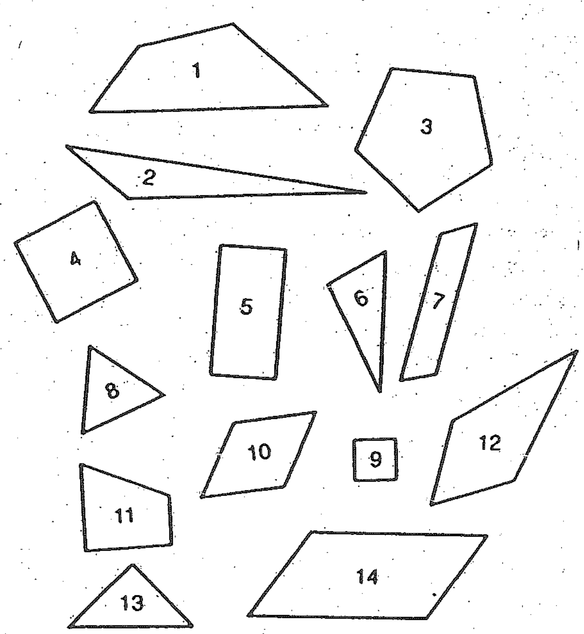
The van Hieles (7, 10, 14, 44) also proposed five instructional phases to guide students' progression from one level to the next on a topic: (a) *information* (two-way conversation between teacher and pupils to acquaint them with the topic and to help the teacher see what they already think about it); (b) *directed orientation* (sequence of activities to engage students actively in exploring the topic); (c) *explicitation* (students become explicitly aware the topic and learn related geometric terminology); (d) *free orientation* (problem solving that requires use and synthesis of the topic); (e) *integration* (students build a summary of the topic and relate this to what they previously learned). Children can cycle through the phases for various topics before attaining the next level of thinking.

Research on the van Hiele Levels

Research on the van Hiele model in primary grades has focused on the usefulness of the levels for characterizing children's thinking and on the impact of instruction on fostering higher levels of thinking (Fig. 9.7). By using clinical interviews involving tasks such as drawing, sorting, and describing shapes, researchers found that youngsters think mainly at the visual level (6, 7, 17). Results confirmed the hierarchical aspect of the levels, but did not uniformly support discontinuity between levels since some children appeared to be in transition between levels 0 and 1, sometimes dealing with shapes visually, and other times in terms of properties (6, 7). Youngsters exhibited difficulties such as (a) recognizing shapes only in standard orientations; (b) incorrect concepts (saying a rectangle has two sides, the vertical ones, and referring to the other two as the "top" and "bottom"); (c) incomplete concepts (not identifying a triangle because it is "too skinny"); (d) imprecise terminology ("even" for parallel). Similar results and difficulties were found for students in grades 5–9, and their levels of thinking varied across different geometric topics (6, 7, 13). Causes for the low level of thinking included lack of instruction on geometry topics and emphasis on the visual level, indicating that we need to assess the amount and quality of the geometry instruction in our classes. Also, the research points out the need to examine closely students' explanations, not just the correctness of their answers, to gain insight into the quality of their thinking and their difficulties.

A brighter picture of how primary students can think is seen in several geometry "teaching experiments." Findings of twenty coordinated studies of geometry learning in grades 2–9 provided "overwhelming evidence . . . that children can learn geo-

FIGURE 9.7 Assessing students' level of thinking



Investigate how your students think about shapes. Show a page of geometric figures or cutout shapes and ask, "which are triangles? . . . rectangles? . . . squares?" Then ask "why" and "how would you tell someone what to look for if he had to pick out all the triangles . . . ." You might repeat such questioning after your instruction to see if your students' thinking has changed.

metrical ideas if they are presented in an appropriate manner" (25, p. 26). Greatest gains were achieved in grades 3–5 when instruction emphasized children's making shapes, finding examples in the environment, and examining shapes in terms of their properties. Attainment of the descriptive-analytic level was reported for Soviet pupils in grades 1–4 (corresponding to our grades 3–6) in a curriculum experiment during the 1960s. The Soviet primary program had been revised to reflect the van Hiele levels and to provide a more diversified, continuous treatment of geometry (33).

Progress toward the descriptive-analytic level has been reported for younger children (grade 1) in a teaching experiment on quadrilaterals (17). Lessons involved only a little work at the visual level and instead stressed relationships between parts of shapes and properties of shapes. Important factors in the instruction included children's use of models; activities with guiding questions to stimulate thinking ("Let's play detective with these clues . . . what could it be? . . . draw the shape in your mind"); repetition and review, in particular of language for parts of shapes and properties; and a general-to-specific sequence of topics (quadrilaterals to rectangle-quadrilaterals to square-quadrilaterals) to help children think about subclass relationships between shapes.

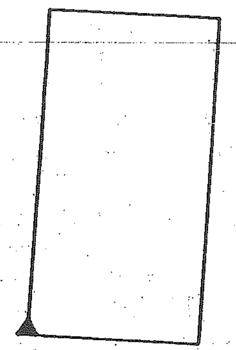
"Logo" is a programming language that can be taught to young children. One aspect of Logo involves creating directions for moving a graphic turtle that leaves a trail of its path on the computer screen. Working with Logo on activities designed to foster thinking about shapes enhanced third and fourth graders' understanding of geometric concepts and also helped them make a transition from the visual level to the descriptive-analytic level (7, 8, 19, 20). Creating a Logo procedure to draw a shape requires children to analyze a shape in terms of its parts (angles, sides) and to reflect on how they can be put together. Children come to view shapes in terms of actions (90-degree turns) to construct them, and as a result, become explicitly aware of those action-based conceptualizations. Children also establish relationships between shapes—for example, realizing that "squares are special rectangles" by observing that their Logo procedure for rectangle (with sides set equal) can make squares. Features of these Logo environments included carefully crafted Logo activities that were coordinated with the math curriculum, cooperative problem solving, and use of multiple representations of concepts (drawings, list of properties, figures on the screen, Logo procedures), which students were encouraged to interrelate (20). This line of research promises to generate exciting developments in the future, such as enriched Logo-based geometry curricula (1) and software designed to guide children to explore topics and to develop more expert understandings (20).

These instructional successes in the primary grades and others in grades 5–9 (7, 13, 25) demonstrate the effectiveness of instruction designed to foster transition to higher levels of thinking. Although additional research is needed, especially teaching experiments that embody the van Hiele phases, the research offers many ideas for classroom practice (Fig. 9.8). First, assessment tasks and instructional activities from these studies can be adapted for use with our students.

- Sorting Shapes. Ask children to put cutout shapes into "families" such as the Right Family (all right angles) or the Four-Siders. Encourage them to find different ways to do this, to invent appropriate family names, and to use informal language of properties.

Drawing shapes with LOGO

Forward 100
Right 90
Forward 50
Right 90
Forward 100
Right 90
Forward 50
Right 90

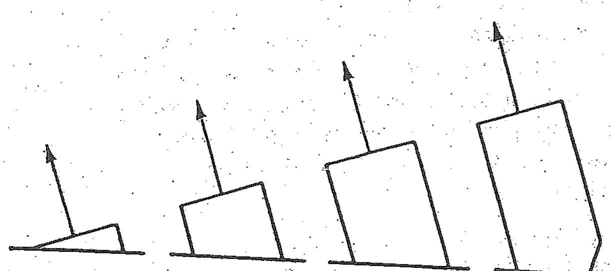


- How might a student modify these commands for a rectangle to draw a square?
- What commands might the student change to draw a rectangle twice as large?

- Shape Detective. Turn over property cards one at a time and ask children to draw what shape the clues specify at each stage and tell when they are sure they can identify the shape (Fig. 9.9). At the end, you might ask which clues were necessary. Also, ask students to create clue puzzles for "geometry sleuth" classmates.
- Shape Paths. Children use Logo-like directions to tell how to draw a shape or walk a path to certain locations (To Library: ahead 10 steps, turn right, ahead 60 steps . . .).

Recommendations regarding the development of geometric language, particularly at the descriptive-analytic level, include: (a) relate unfamiliar new words to similar familiar ones (triangle, tricycle), (b) show explicitly that a term (face, side) has a different meaning in mathematics than it does in everyday usage; (c) use cooperative learning, which promotes both receptive and expressive use of language; and (d) pe-

FIGURE 9.8 Uncover-the-shape task

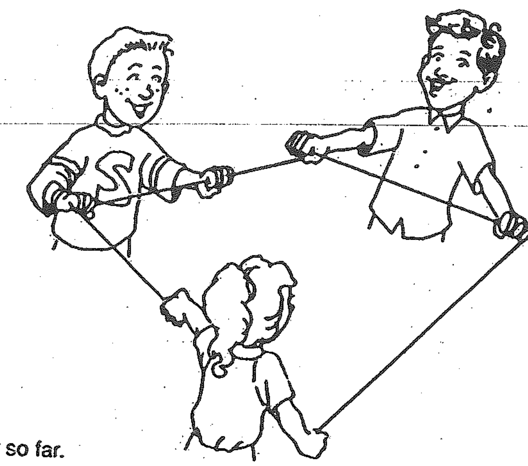


Uncover the **Mystery Shape** in stages. Ask what shape it could be and could not be and "why" at each stage.

FIGURE 9.9 Uncovering-property-clue-cards activity

What do I look like? Who am I?

- #1 for angles
- #2 four angles
- #3 one obtuse angle
- #4 two right angles
- Clue #5



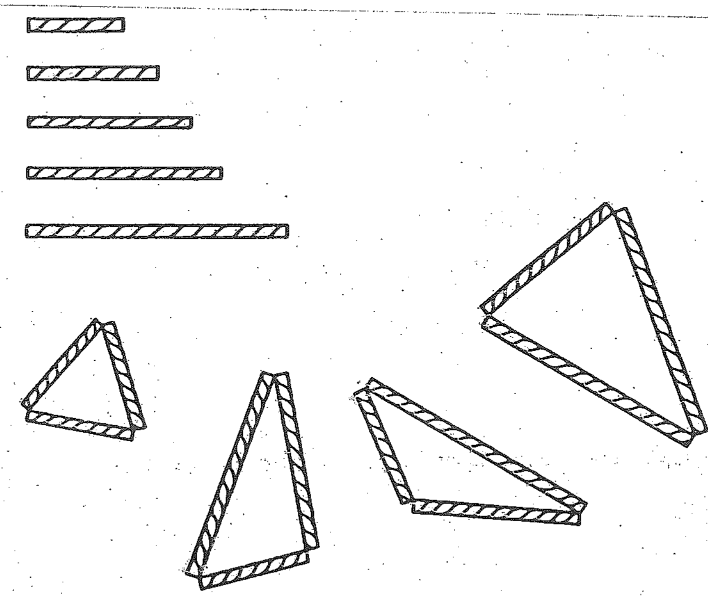
Four clue cards have been turned over so far.
Can we tell what shape it is?
Do we need all the clues 1-4? Explain.
What could clue 5 be?
Such questions can foster desciptive-analytic thinking about shapes and also lead to the next level of informal deduction.

riodically review concepts and related terminology. Attention should also be given to logical and metacognitive language (13). Encourage careful use of quantifiers (all, some, none) and phrases for generalizations ("*a* square has . . ." to mean "*all* squares have . . . "). Words like "look at" and "identify" usually convey the visual level, while "property," "discover," and "explain" connote higher levels. Thoughtful use of such metacognitive words by you and your students can help them to realize the kind of thinking that is expected and to monitor their thinking relative to the expectations.

Recommendations can also be derived from research that analyzed geometry curricula in terms of the van Hiele model (13, 33). An analysis of the geometry material in three major USA text series (K–8) indicated that the K–4 geometry lessons (aim, expository material, exercises, test questions) dealt almost exclusively with the visual level (13). The cumulative effect of such low level curricula is to limit, or even impede, the development of children's geometric thinking during grades K–4. Little wonder that many students encounter difficulty in middle school and secondary school geometry, which presumes a sound apprenticeship in visual and descriptive-analytic thinking during primary grades. Clearly, we should not rely on textbooks alone for geometry instruction. We need to:

- Analyze the expository and exercise material and related suggestions in the teacher's guide before doing lessons and "fill in" gaps, such as reminding students to "explain why" when doing exercises in the text.

Making triangles with straws cut in five lengths



Imagine children working in groups to make triangles and investigating questions such as:

- Can we make a triangle with any three straws? Make a conjecture about when three straws will not work.
- How are two triangles alike? different?
- How can we sort the triangles?
- Where in our environment do we find these different types of triangles?
- Such purposeful questions stimulate thinking at the analytic-descriptive level. What other questions might we pose?

- Implement "purposeful" activities that provide a natural transition toward a higher level.
- Spiral activities on geometry topics thoughout the curriculum to provide a more continuous experience with those topics.
- Include questions during class discussions and on tests to assess thinking above the visual level. We probably will need to talk to children to assess the quality of their thinking, especially youngsters who cannot adequately express their ideas in writing.

Concept Learning in Geometry

How well do primary students understand such geometric concepts as angle, rectangle, cube? How might your students show they understand them? These are important questions since the geometry strand for K–4 programs contains a host of concepts involving two- and three-dimensional shapes. More generally, the *Standards* states that "the K–4 curriculum should be conceptually oriented" (28, p. 17). However, NAEP testing (18) indicates that many students demonstrate a lack of understanding of underlying concepts in mathematics. Surveys of classroom practice show "a heavy emphasis on skill development and slight attention to concepts" (31, p.11). Also, children exhibit various types of misconceptions such as under-generalization, which can occur because they included irrelevant characteristics; over-generalization, which can occur because some key properties are omitted; and languange-related misconceptions ("diagonal" means slanty). Research on concept learning sheds some light on how children form concepts (and misconceptions) with implications for ways to teach geometric concepts and also concepts in other areas of mathematics (even/odd, factor) and in other subjects.

One way children form a concept is to begin with a few cases and by averaging their features develop an "average representation" or prototype, which they use for categorizing new examples (34). Children may form prototypes that include extraneous or even erroneous features that can lead to misconceptions (7, 13, 41) such as thinking of a shape only in terms of cases in standard orientation, as usually found in the everyday world. However, beginning with "best examples," that is, "clear cases demonstrating the variation of the concept's attributes" (41, p. 281), can help develop correct concepts (Fig. 9.10). We need to provide "best examples" that are rich in imagery of familiar everyday objects and manipulative models that children have worked with. Research (20) indicates that children should develop multiple representations of concepts (everyday objects, manipulatives, diagrams, verbal definitions). Some children encode information better in a verbal format than visual, others vice-versa. Also, they should link representations for a concept, such as by drawing examples and giving a list of properties.

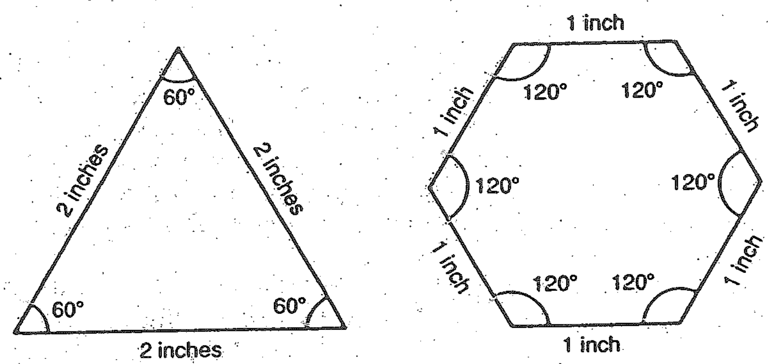
Primary pupils are better able to learn geometry concepts when they handle models and use diagrams (7, 25, 32). They should experience both region/surface models and side/edge models (Fig. 9.11). Children may find some aspects of shapes easier to grasp with one type of model than another. Here too they can connect representations for a concept—for example, interrelating angle as a "wedge," made with two fastened straws that open, a turn of their body, or a Logo turtle turn.

It is important to present both examples and nonexamples of a concept (7, 28, 41) as in the concept card (Fig. 9.12). Nonexamples have proved more effective than examples for "difficult" concepts and when familiar prototypes for a concept frequently have irrelevant features. Nonexamples should vary all irrelevant features. Carefully chosen nonexamples help children eliminate irrelevant features and identify critical ones.

The concept-card approach helps children learn how to formulate a correct definition. Initally children can use their own language for definitions, although it will be imprecise at times, and then through class discussion formulate a working verbalization using geometric terms. Youngsters may need to touch parts of shapes or concrete models when explaining verbally. Children sometimes memorize verbal definitions and can spout them with ease, so we are cautioned not to rely solely on verbal definitions to see if children understand a concept. Ask them to draw examples and nonexamples, or explain which cases are or are not examples and why.

FIGURE 9.10 Best examples

There are many examples of **regular polygons**.
Here are some best examples.



Fourth graders used these best examples to learn about regular polygons (41, p. L82). What other examples could we have used? We might investigate which examples work best for our students.

FIGURE 9.11 Region/surface models and side/edge models

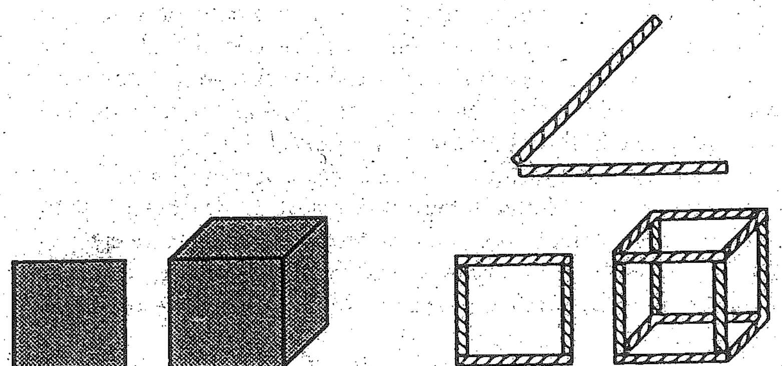
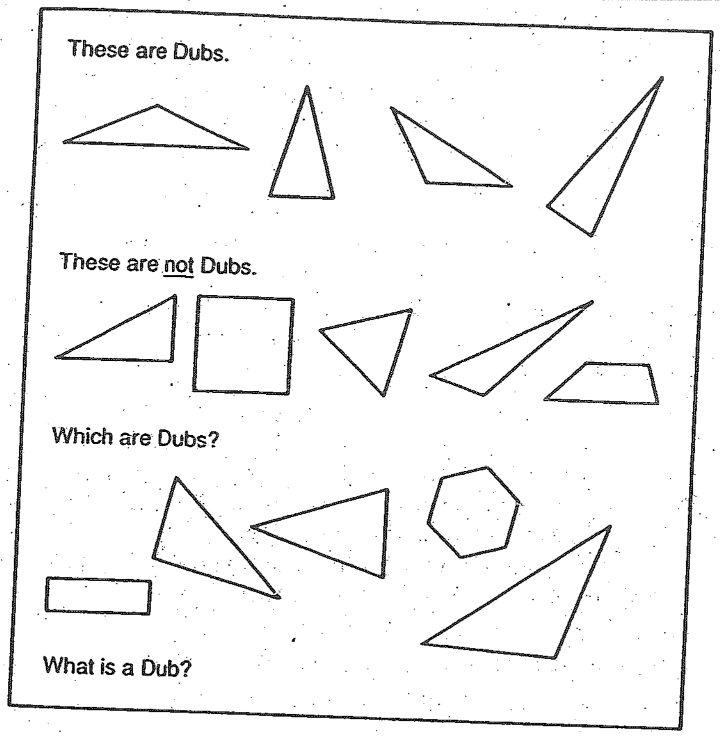


FIGURE 9.12 Concept card with examples and nonexamples



Children can identify examples of a concept by thinking in terms of prototypes, which has advantages (speed in recognizing familiar examples) and disadvantages (misidentification of unusual cases). Another way is to use a rule or definition for the concept, that is, to think in terms of key properties of the concept. Rule-based concepts are powerful in that they enable one to identify and generate examples and nonexamples and also to deduce conclusions from them (34).

One way to express a rule is by a verbal definition or list of properties called a "declarative specification" of the concept—that is, a statement of "what it is" in terms of its characterizing features (20, 34). Most textbook definitions are this type. Another way is through a "procedural specification," which enables you to construct or identify examples the concept. A child might give a procedural description for "square" by telling how to make it with manipulatives ("take four sticks the same size, put two like this for a square corner, and then . . .") or by a Logo procedure. Children can use one type of specification to understand another—for example, a third grader who came to understand her textbook's declarative definition for square by thinking about her procedural Logo-notion.

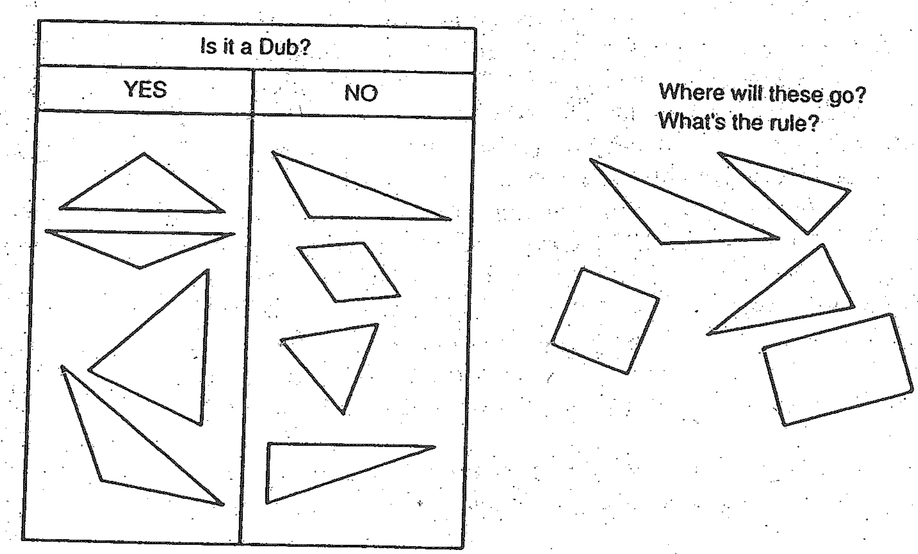
We can incorporate examples and nonexamples, best examples, and different representations of concepts into discovery and expository lessons for introducing geometric concepts.

Formulating a declarative definition from a procedural one
Confused by her text's definition for square (a quadrilateral with right angles and equal sides), Jenny reflects on her Logo experiences. "I need four FORWARD 100 and four RIGHT 90 . . . so oh . . . four sides . . . equal sides . . . right angles . . . yeah, I got it now." How might Jenny think about rectangles? Might she think of squares as special rectangles?

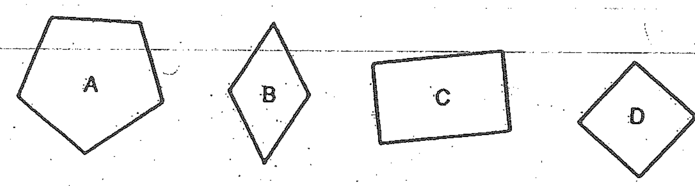
Discovery Lessons. Present a concept card on the board and guide children to formulate and test a rule for the shape. One variation is to present the examples and nonexamples one at a time (on the board or via computer "guess-my-rule" software) and challenge children to predict where they go and why (Fig. 9.13). Another variation is to have children work individually or in small groups on concept cards in a learning center and later share their definitions.

Expository Lessons. First present two "best examples" (leaving them out for viewing), next give the critical features of the concept as they relate to the examples, and then provide questions structured to have children check each critical property for several cases. This approach proved more effective for teaching third and fourth graders geometric concepts than did those not having "best examples" or not using examples and nonexamples (41), suggesting why the expository material in many textbooks may not suffice for developing concepts—namely, examples are seldom "best examples," nonexamples are rare, and exercises usually ask only for identification, not "why."

FIGURE 9.13 Guess-my-rule sort of examples and nonexamples



Checklist of critical properties



Answer these questions about shapes A, B, C, D.

Is it a polygon?	YES NO
Does it have all sides of equal length?	YES NO
Does it have all angles equal?	YES NO
Is it a regular polygon?	YES NO

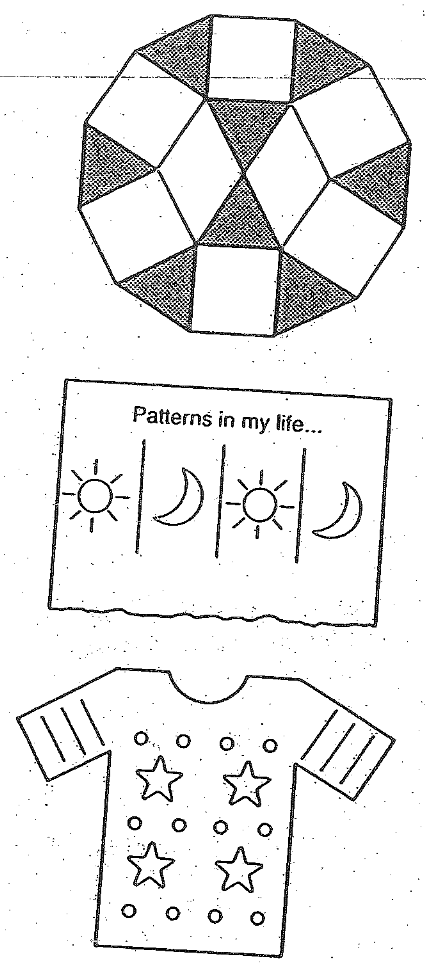
Early Childhood Perspectives

Perspectives on teaching geometry in the primary grades can also be derived from general research, theory, and practice in early childhood education (16, 27). One guiding principle is that instruction should involve "the whole child" and his or her cognitive, affective, social, and physical needs and characteristics. Goals (16) fall into four areas: *knowledge, skills, dispositions*, and *feelings*. Too often pressures "to cover the curriculum" have a negative effect on these goals. A second principle is that instruction should be based on how children learn, namely: by doing, reflecting on their actions, and sharing their ideas with classmates and the teacher. Third, instruction should feature a variety of methods (play, learning center, projects, direct teaching). Teachers are cautioned that methods dominated by workpages can have a cumulative negative effect over a year or two, stultifying youngsters' disposition to learn. Finally, children need opportunities to engage in activities that call for extension, elaboration, and continuation of ideas and for sustained effort over time (days and weeks). A steady diet of tasks that children can finish quickly with little thought can have a delayed, detrimental effect on their desire to learn more and to explore (16).

Below are samples of geometry activities, projects, and environments that embody these kinds of principles. In reading them, think about which research perspectives on teaching and learning geometry they also reflect and how you might adapt them for your classroom.

Patterns Around Us. Children copy, extend, and create patterns with manipulatives such as pattern blocks, which leads to a hunt for geometric patterns around them (wallpaper, floor tiling) and patterns in their own lives (Fig. 9.14). Youngsters decide to create their own patterns with poster paints and geometric potato prints. Experiences with children's books on geometry (42) prompts some to make their own

FIGURE 9.14 Patterns and geometric designs



shape-books; others fashion colorful designs that become part of a class "patterns" wall mural. A culminating activity might be to make functional "pattern" objects such as designer T-shirts, bracelets, or soup can pencil holders.

Let's Build It. Children design and build some large construction like a geodesic greenhouse, big-box learning center for their classroom, or model school bus (16), integrating skills in geometry, measurement, estimation, and construction. A discussion of the properties of shapes and their uses in constructions can lead to an exploration of relationships between "form" and "function."

The Block Corner. Children enjoy building blocks. Their teacher capitalizes on this interest and creates block task cards (Fig. 9.15) that guide children to discover relationships between blocks (two s make a "circle" wheel; four s make a long block). As variations, the teacher creates similar task cards involving "found objects" such as woodscraps and cereal boxes and Water-blocs (soft 3-D shapes that adhere when wet) for use at the water table.

Triangle Investigations. Using the van Hiele phases to design a mini-unit on triangles, some teachers work together to prepare activities involving geoboards and straws and pipe-cleaner connectors. In their classrooms colorful displays show that children have explored triangle designs on geoboards and dot paper (*information*). Next children made triangles with straws cut in various lengths, sorted them (0, 2, 3 equal sides), learned names (scalene, isosceles, equilateral), and discussed why those names were appropriate (*orientation* and *explicitation*). Now they begin to work in groups on challenging geoboard problems (*free orientation*). Later, they will make mobiles of straw shapes to show the relationship between triangles and polygons that they had investigated previously (*intetgration*) (Fig. 9.16). Still later teachers plan a similar cycle of activities on quadrilaterals.

A Model City. Integrating geometry with children's study of their neighborhood or city, the class constructs a "model city." Working in committees (Bureau of Streets), they design and build a model city. Functional aspects of geometry (parallel and perpendicular lines, angles for streets; types of solids for buildings) are a natural part of this environmental geometry project. Children might also explore the "city" of American Indians or early settlers who lived in their area, noting geometric designs in their clothing, crafts, and living quarters (40).

FIGURE 9.15 Activity cards involving building blocks

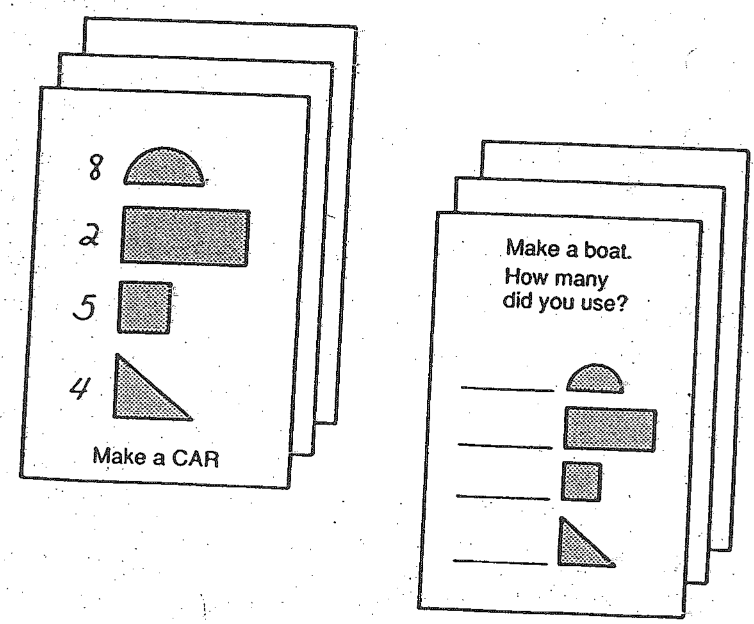
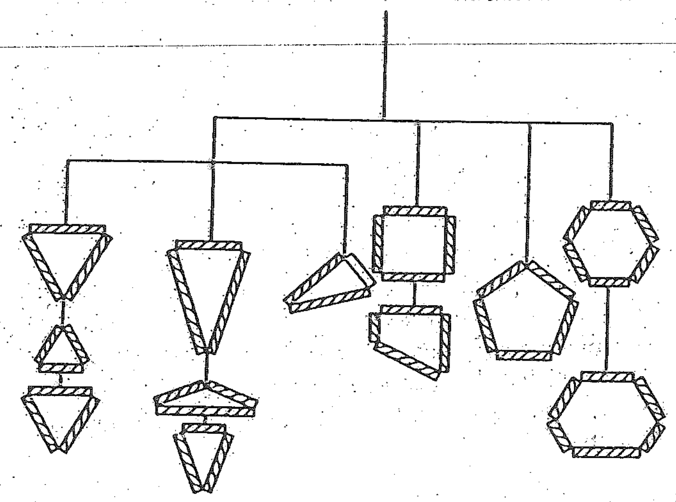
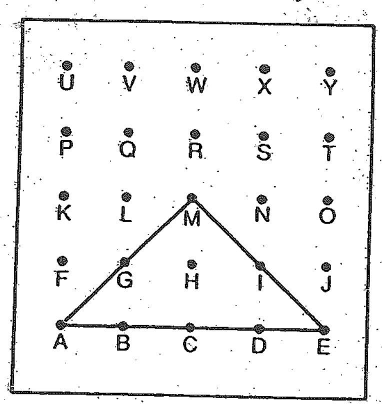


FIGURE 9.16 Mobile of shapes for families of triangles and polygons



Free orientation activity about triangles on a geoboard

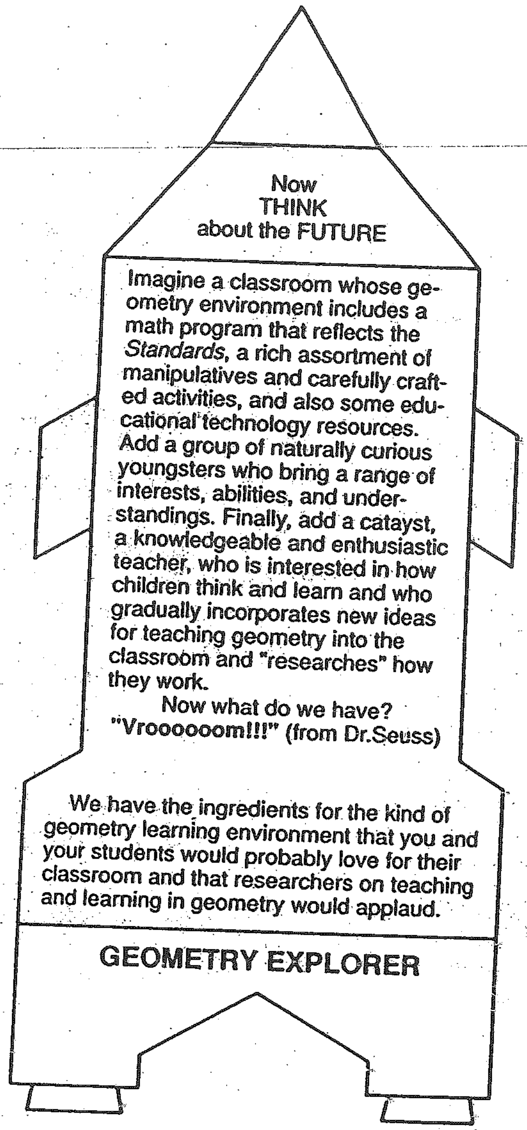


- How many isosceles triangles are there with side AE? with a corner at A? at M?
- Explore: What if we pick other points for a corner?

Conclusion

The research provides us with perspectives for reflecting on the geometry learning environment in our classroom today and also for planning for the future. It challenges us to provide a richer, more continuous activity-oriented program in geometry. It also offers principles and effective approaches for classroom practice. Finally, as we

FIGURE 9.17 Geometry explorer



help youngsters become explorers of geometry, the research invites us to raise and address questions about ways to improve the teaching and learning of geometry in our classrooms.

Looking Ahead . . .

This chapter has messages for several audiences. Teachers need to think about the geometry they *should* teach and how they can use research-based ideas in their teaching. Those responsible for pre- and in-service training are challenged to provide experiences that develop teachers' geometric thinking, enthusiasm for geometry, and sensitivity to children's thinking about geometry. Researchers need to explore the nature of children's geometric thinking and ways to develop it in a variety of primary grade settings. Finally, we all need to share research findings and to work together to translate them into classroom practice.

David J. Fuys

Reflecting on the geometry learning environment in my classroom, I wonder about ways to create a more global and integrated geometry program. I am concerned about using time wisely and how to "fit in" all the important topics. I need to evaluate the activities my students currently use and identify additional resources for a more stimulating and enriched program. I wonder where I can find such resources, and how I can implement them using research-based approaches. Finally, I feel the need to interact with colleagues about our concerns, resources, and innovations involving the teaching and learning of geometry in our classes.

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